

A lifting functor for toric sheaves

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Abstract

For a variety X which admits a Cox ring, we introduce a functor from the category of quasi-coherent sheaves on X to the category of graded modules over the homogeneous coordinate ring of X . We show that this functor is right adjoint to the sheafification functor and therefore left-exact. Moreover, we show that this functor preserves torsion-freeness and reflexivity. For the case of toric sheaves, we give a combinatorial characterization of its right derived functors in terms of certain right derived limit functors.

1 Introduction

Consider an affine normal variety $W = \operatorname{Spec}(S)$ over an algebraically closed field K , G a diagonalizable group scheme which acts on W , and $H \subseteq G$ a closed subgroup scheme. We denote T the quotient of diagonalizable groups schemes G/H . Moreover, we assume the following.

- There exists a Zariski-open G -invariant subset \hat{X} of W such that a good quotient $X = \hat{X} // H$ exists. We denote $\pi : \hat{X} \rightarrow X$ the corresponding projection.
- X admits an affine T -invariant open covering (this is automatic if T is a torus, see [Sum74]).
- The complement $Z = W \setminus \hat{X}$ has codimension at least 2.

The actions of G and H on W induce gradings on S by the character groups $X(G)$ and $X(H)$, respectively, which are compatible via the surjection $X(G) \twoheadrightarrow X(H)$. With this, we require moreover the following.

- $X(H) \cong A_{d-1}(X)$, the divisor class group, and for suitable representatives $D_\chi \in A_{d-1}$ there exists an isomorphism of S_0 -modules $S \cong \bigoplus_{\chi \in X(H)} \Gamma(\mathcal{O}(D_\chi))$ which is compatible with the $X(H)$ -grading of S . In particular, the latter carries an induced ring structure.

These conditions essentially imply that X is a variety which admits a Cox ring, where we admit possibly some further action by a diagonalizable group scheme. In particular, this class of varieties includes the Mori dream spaces. The main application we have in mind is the case where T is a torus and X a toric variety such that S is the associated homogeneous coordinate ring as defined in [Cox95]. It was shown in [PT10, §6] that by taking local invariants we obtain an exact and essentially surjective functor which maps an $X(G)$ -graded S -module E to a T -equivariant quasi-coherent sheaf \tilde{E} on X , the so-called *sheafification functor*. Conversely, there is a functor from the category of T -equivariant sheaves on X to the category of $X(G)$ -graded S -modules, mapping a quasi-coherent sheaf \mathcal{E} to a $X(G)$ -graded S -module $\Gamma_* \mathcal{E} := \Gamma(\hat{X}, \pi^* \mathcal{E})$. This functor is right inverse to the sheafification functor, i.e., we have $\widetilde{\Gamma_* \mathcal{E}} \cong \mathcal{E}$ for any \mathcal{E} .

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However, the functor Γ_* in many cases is not very well behaved. So it usually does not preserve properties such as torsion-freeness and reflexivity. Also, by being the composition of the right-exact functor π^* with the left-exact global section functor, Γ_* does not have any exactness properties. In general, Γ_* is right-exact if $\hat{X} = W$ (and thus X is affine) and it is left-exact if π is a flat morphism.

The aim of this note is to construct an alternative functor to Γ_* , which we are going to call the *lifting functor*, which maps a quasi-coherent T -equivariant sheaf \mathcal{E} to an $X(G)$ -graded S -module $\hat{\mathcal{E}}$. We will show that the lifting functor has the following two general properties:

1. The lifting functor is right adjoint the sheafification functor and therefore left-exact (Theorem 3.8).
2. Lifting preserves torsion-freeness and reflexivity. For torsion free sheaves it preserves coherence (Theorem 4.4).

The lifting functor is an offspring of recent work on toric sheaves [Per11]. Assume that X is a toric variety and \hat{X} the standard quotient presentation as in [Cox95]. By results of Klyachko [Kly90], [Kly91], any coherent reflexive T -equivariant sheaf \mathcal{E} can be described by a finite-dimensional vector space together with a family of filtrations parameterized by the rays of the fan associated to X . In order to represent \mathcal{E} by an appropriate \mathbb{Z}^n -graded module over the homogeneous coordinate ring, it is a rather straightforward observation that, rather than taking $\Gamma_*\mathcal{E}$, we can choose a reflexive sheaf which is associated to precisely the same filtrations as \mathcal{E} (this is possible because there is a one-to-one correspondence among the rays of the fans associated to X and \hat{X} , respectively). Our results show that this ad-hoc observation indeed has a functorial interpretation. In Section 5 we will see that the lifting functor has moreover a very nice interpretation in the combinatorial setting of [Per11].

2 Preliminaries

2.1. Let A be any abelian group, S a A -graded K -algebra, and E an A -graded S -module. Then $E \cong \bigoplus_{\alpha \in A} E_\alpha$ and for any $\beta \in A$ we denote $E(\beta) = \bigoplus_{\alpha \in A} E_{\alpha+\beta}$ the *degree shift* of E by β .

2.2. For any two S -modules E and F , The tensor product $E \otimes_R F$ can be A -graded as follows. Consider first the K -vector space $E \otimes_K F$ and set $(E \otimes_K F)_\alpha = \bigoplus_{\beta \in A} (E_\beta \otimes_K F_{\alpha-\beta})$. Then for $\alpha \in A$ we form $(E \otimes_R F)_\alpha$ as the quotient of $(E \otimes_K F)_\alpha$ by the subvector space generated by $re \otimes f - e \otimes rf$ for $e \in E, f \in F, r \in R$. Note that $E(\alpha) \otimes_R F \cong E \otimes_R F(\alpha) \cong (E \otimes_R F)(\alpha)$.

2.3. For any A -graded S -modules E, F , the *graded* version of $\text{Hom}_S(E, F)$ by definition is given by

$$\text{HOM}_S^A(E, F) := \bigoplus_{\alpha \in A} \text{Hom}_S^A(E, F(\alpha)),$$

where $\text{Hom}_S^A(E, F(\alpha)) = \{f \in \text{Hom}_S(E, F) \mid f(E_\beta) \subseteq F_{\beta+\alpha} \text{ for every } \beta \in A\}$. We can consider in a natural way $\text{HOM}_S^A(E, F)$ as a subset of $\text{Hom}_R(E, F)$. Moreover, within the graded setting, HOM_S^A satisfies the same general functorial properties as the standard Hom (see [Nv04, §2]). Note that when we speak of the category of A -graded modules, then the set of morphisms between modules E and F is given by $\text{Hom}_S^A(E, F)$ and *not* by $\text{HOM}_S^A(E, F)$.

2.4. We will deal with three gradings, given by the character groups $X(T)$, $X(G)$, and $X(H)$, respectively. Any given $X(G)$ -graded ring S carries an $X(H)$ -grading as well via the surjection $X(G) \twoheadrightarrow X(H)$. To distinguish between these two gradings, we write the homogeneous components $S_{(\alpha)}$ and S_χ for the $X(H)$ - and the $X(G)$ -grading, respectively, where $\alpha \in X(H)$ and $\chi \in X(G)$. For $\chi \in X(G)$ we may also write $S_{(\chi)}$ for the $X(H)$ -homogeneous component determined by the image of χ in $X(H)$. Then $S_{(\chi)}$ has a natural $X(T)$ -grading which is given

by $S_{(\chi)} \cong \bigoplus_{\eta \in X(T)} (S_{(\chi)})_{\eta}$ with $(S_{(\chi)})_{\eta} = S_{\chi+\eta}$. We use the same conventions for $X(G)$ - and $X(H)$ -graded S -modules.

2.5. For any $X(G)$ -graded S -modules E, F , we have the two graded modules $\mathrm{HOM}_S^{X(G)}(F, E)$ and $\mathrm{HOM}_S^{X(H)}(F, E)$, together with the natural sequence of inclusions

$$\mathrm{HOM}_S^{X(G)}(F, E) \subseteq \mathrm{HOM}_S^{X(H)}(F, E) \subseteq \mathrm{HOM}_S(F, E)$$

(which even satisfy certain topological properties, [Nv04, §2.4]).

2.6. The $X(H)$ -invariant subring $R = S_{(0)}$ is automatically $X(T)$ -graded. It is also $X(G)$ -graded by trivial extension, i.e., we set $R_{\chi} = 0$ for every $\chi \in X(G) \setminus X(T)$. Likewise, every $X(T)$ -graded R -module can be given an $X(G)$ -grading.

2.7. With the notation as in 2.3, note that we have $\mathrm{HOM}_S^A(F, E) = \bigoplus_{\alpha \in A} \mathrm{Hom}_S^A(F, E(\alpha)) = \bigoplus_{\alpha \in A} \mathrm{Hom}_S^A(F(-\alpha), E)$. However, in order to avoid some cumbersome signs, we will usually write expressions like $\hat{E} = \bigoplus_{\alpha \in A} \mathrm{Hom}_S^A(S(\alpha), E)$, where it is understood that the proper grading is given by $(\hat{E})_{\alpha} = \mathrm{Hom}_S^A(S(-\alpha), E)$.

2.8. The sheafification functor as defined in [PT10] maps an $X(H)$ -graded (respectively $X(G)$ -graded) S -module E to a quasi-coherent sheaf \hat{E} over X as follows. Let open affine covers $\{U_i = \mathrm{Spec}(R_i)\}_{i \in I}$ and $\{\hat{U}_i = \pi^{-1}(U_i) = \mathrm{Spec}(S_i)\}_{i \in I}$ on X and \hat{X} , respectively, be given, such that $U_i = \hat{U}_i // H$ (both covers can be chosen T and G -invariant, respectively). Then $R_i = S_i^H = (S_i)_{(0)}$ for every $i \in I$ and we can associate to every U_i the R_i -module $\Gamma(\hat{U}_i, E)_{(0)}$, where by abuse of notation we identify E with its associated quasi-coherent sheaf over W . These glue naturally to give a quasi-coherent sheaf of \mathcal{O}_X -modules. Moreover, if the U_i are chosen T -invariant, then the R_i are $X(T)$ -graded, and both the R_i and S_i are $X(G)$ -graded by 2.6. In this case, \mathcal{E} has also an induced T -equivariant structure.

3 The right adjoint

For a given morphism of schemes $f : U \rightarrow V$ and a quasi-coherent sheaf \mathcal{F} on V , it is standard to define the pullback $f^*\mathcal{F}$ as $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_V} \mathcal{O}_U$. This defines a right-exact functor from the category of (quasi-)coherent \mathcal{O}_V -modules to the category of (quasi-)coherent \mathcal{O}_U -modules. However, this is not the only conceivable way to define a pull-back functor; instead, one could consider the sheaf

$$f^{\wedge}\mathcal{F} := \mathcal{H}om_{f^{-1}\mathcal{O}_V}(\mathcal{O}_U, f^{-1}\mathcal{F}).$$

Clearly, f^{\wedge} is a left-exact functor from the category of quasi-coherent \mathcal{O}_V -modules to the category of quasi-coherent \mathcal{O}_U -modules. In the affine case, i.e., $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(B)$ for some commutative rings A, B , and \mathcal{F} the sheaf corresponding to an B -module F , $f^{\wedge}\mathcal{F}$ corresponds to the module $\mathrm{Hom}_B(A, F)$, where the A -module structure is given by $(rg)(r') = g(rr')$ for $r, r' \in R$ and $g \in \mathrm{Hom}_B(A, F)$.

However, the following example shows that f^{\wedge} in general will behave quite pathological.

Example 3.1. Assume $R = F = K$ and $T = K[x]$. Then we have isomorphisms of K -vector spaces

$$\mathrm{Hom}_K(K[x], K) \cong \mathrm{Hom}_K\left(\bigoplus_{i \geq 0} K, K\right) \cong \prod_{i \geq 0} \mathrm{Hom}_K(K, K) \cong \prod_{i \geq 0} K.$$

That is, from a one-dimensional K -vector space we have created a $K[x]$ -module which has torsion elements and no countable generating set.

We will see that one can define a better behaved graded version of this pull-back. Under our general assumptions on X and \hat{X} , let $\{U_i\}_{i \in I}$ and $\{\hat{U}_i = \pi^{-1}(U_i)\}_{i \in I}$ be affine T - and G -invariant covers, respectively, as in 2.8. By the general properties of good quotients, the \hat{U}_i form an affine open covering of \hat{X} such that $U_i = \hat{U}_i // H$ for every $i \in I$. We denote $U_i = \text{Spec}(R_i)$ and $\hat{U}_i = \text{Spec}(S_i)$; then $R_i = S_i^H$ for every $i \in I$. Moreover, both the R_i and S_i are $X(G)$ -graded by 2.6. For any character (and thus divisor class of X) $\alpha \in X(H)$, there is naturally associated the module $\mathcal{O}(\alpha) \cong \widehat{S(\alpha)}$, which is reflexive and of rank one. This module is a distinguished representative for the isomorphism class of such sheaves associated to the class α . Similarly, if we choose some $\chi \in X(G)$ which maps to α via the surjection $X(G) \twoheadrightarrow X(H)$, we obtain an induced T -equivariant structure on $\mathcal{O}(\alpha)$, which we denote by $\mathcal{O}(\chi) \cong \widehat{S(\chi)}$.

Definition 3.2. Let \mathcal{E} be a T -equivariant quasi-coherent sheaf on X . Then we set

$$\mathcal{E}^H := \bigoplus_{\alpha \in X(H)} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}(\alpha), \mathcal{E}).$$

and

$$\mathcal{E}^G := \bigoplus_{\chi \in X(G)} \mathcal{H}om_{\mathcal{O}_X}^T(\mathcal{O}(\chi), \mathcal{E}).$$

We first show the following.

Proposition 3.3. Let \mathcal{E} be a T -equivariant quasi-coherent sheaf on X .

- (i) Both \mathcal{E}^H and \mathcal{E}^G are quasi-coherent subsheaves of $\pi^* \mathcal{E}$, and $\mathcal{E}^H \cong \mathcal{E}^G$ as $\mathcal{O}_{\hat{X}}$ -modules.
- (ii) \mathcal{O}_X^H (and therefore \mathcal{O}_X^G) is isomorphic to $\mathcal{O}_{\hat{X}}$.
- (iii) If $\Gamma(U_i, \mathcal{E})$ is a first syzygy module for every i , then so is $\Gamma(\hat{U}_i, \mathcal{E}^H)$.
- (iv) If \mathcal{E} is coherent and torsion free, then \mathcal{E}^H and \mathcal{E}^G are coherent and torsion free as well.

Proof. (i) First note that for every $\chi \in X(G)$ which maps to $\alpha \in X(H)$, we have a natural inclusion of sheaves of K -vector spaces $\phi_\chi : \mathcal{H}om_{\mathcal{O}_X}^T(\mathcal{O}(\chi), \mathcal{E}) \hookrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}(\alpha), \mathcal{E})$. Summing over all such characters, we get a map of sheaves $\sum_{\eta \in X(T)} \phi_{\chi+\eta} : \bigoplus_{\eta \in X(T)} \mathcal{H}om_{\mathcal{O}_X}^T(\mathcal{O}(\chi+\eta), \mathcal{E}) \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}(\alpha), \mathcal{E})$. Locally, we denote $E_i := \Gamma(U_i, \mathcal{E})$ for every U_i and this map translates to an isomorphism of R_i -modules $\bigoplus_{\eta \in X(T)} \text{Hom}_{R_i}^{X(T)}((S_i)_{(\chi+\eta)}, E_i) \rightarrow \text{Hom}_{R_i}^{X(T)}((S_i)_{(\alpha)}, E_i)$. Because $(S_i)_{(\alpha)}$ is a finitely generated R_i -module by our general assumptions, the latter is isomorphic to $\text{Hom}_{R_i}((S_i)_{(\alpha)}, E_i)$ (see [Nv04, Cor. 2.4.4]). So, ϕ_α is indeed an isomorphism and by summing over all $\chi \in X(G)$, we get an isomorphism $\sum_{\chi \in X(G)} \phi_\chi : \mathcal{E}^G \rightarrow \mathcal{E}^H$. Now, $\Gamma(\hat{U}_i, \mathcal{E}^H) \cong \bigoplus_{\alpha \in X(H)} \text{Hom}_{R_i}((S_i)_{(\alpha)}, E_i)$ and therefore \mathcal{E}^H (and thus \mathcal{E}^G) is quasi-coherent. Moreover, observe that locally we have $\Gamma(\hat{U}_i, \pi^* \mathcal{E}) \cong \text{Hom}_{R_i}(S_i, E) \cong \text{Hom}_{R_i}(\bigoplus_{\alpha \in X(H)} (S_i)_{(\alpha)}, E_i) \supseteq \bigoplus_{\alpha \in X(H)} \text{Hom}_{R_i}((S_i)_{(\alpha)}, E_i)$, so \mathcal{E}^H (and thus \mathcal{E}^G) indeed is a subsheaf of $\pi^* \mathcal{E}$.

(ii) It suffices to show that for any i the module $\hat{R}_i := \bigoplus_{\alpha \in X(H)} \text{Hom}_{R_i}((S_i)_{(\alpha)}, R_i)$ is naturally isomorphic to S_i . For this, we observe that for every $\alpha \in X(H)$ holds $(\hat{R}_i)_{(\alpha)} = \text{Hom}_{R_i}((S_i)_{(-\alpha)}, R_i) \cong (S_i)_{(\alpha)}$, as the $(S_i)_{(\alpha)}$ are reflexive modules of rank one by our general assumptions. Therefore we have natural isomorphisms $\hat{R}_i \cong \bigoplus_{\alpha \in X(H)} (\hat{R}_i)_{(\alpha)} \cong \bigoplus_{\alpha \in X(H)} (S_i)_{(\alpha)} \cong S_i$.

(iii) By assumption, we can represent E_i as a first syzygy $0 \rightarrow E_i \rightarrow R_i^{\oplus I}$, where I is some index set. Applying $\bigoplus_{\alpha \in X(H)} \text{Hom}_{R_i}((S_i)_{(\alpha)}, \cdot)$ preserves left-exactness and direct sums in the right argument, and so we obtain an exact sequence $0 \rightarrow \hat{E}_i \rightarrow \hat{R}_i^{\oplus I} \cong S_i^{\oplus I}$, where $\hat{E}_i := \bigoplus_{\alpha \in X(H)} \text{Hom}_{R_i}((S_i)_{(\alpha)}, E_i) \cong \Gamma(\hat{U}_i, \mathcal{E}^H)$, and the latter isomorphism follows from (ii).

(iv) It suffices to show that for any i the module $\hat{E}_i := \bigoplus_{\alpha \in X(H)} \text{Hom}_{R_i}((S_i)_{(\alpha)}, E_i)$ is torsion free. Because E_i is by assumption torsion free and finitely generated, it can be represented as a first syzygy module $0 \rightarrow E_i \rightarrow R_i^{n_i}$ for some integer n_i . Applying (iii), we obtain an exact sequence $0 \rightarrow \hat{E}_i \rightarrow S_i^{n_i}$. Hence, \hat{E}_i is finitely generated and torsion free. \square

We come now to our main definition.

Definition 3.4. Let \mathcal{E} be a T -equivariant quasi-coherent sheaf on X . Then we call the $X(G)$ -graded S -module

$$\hat{\mathcal{E}} := \Gamma(\hat{X}, \mathcal{E}^G)$$

the *lifting* of \mathcal{E} .

Remark 3.5. Note that the S -module $\hat{\mathcal{E}}$ carries both a $X(G)$ -grading as well as a $X(H)$ -grading, which are given by

$$\hat{\mathcal{E}} \cong \bigoplus_{\alpha \in X(H)} \text{Hom}_{\mathcal{O}_X}(\mathcal{O}(\alpha), \mathcal{E}) \cong \bigoplus_{\chi \in X(G)} \text{Hom}_{\mathcal{O}_X}^T(\mathcal{O}(\chi), \mathcal{E})$$

(see 2.7 for our convention on the grading). By construction, lifting is functorial and left-exact. Moreover, if X is smooth, then every sheaf of the form $\mathcal{O}(\alpha)$ is invertible and we have natural isomorphisms $\text{Hom}_{\mathcal{O}_X}(\mathcal{O}(\alpha), \mathcal{E}) \cong \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}(-\alpha))$ for every α . (respectively $\text{Hom}_{\mathcal{O}_X}^T(\mathcal{O}(\chi), \mathcal{E}) \cong \Gamma(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}(-\chi))^T$ for every χ). In this case, our lifting functor is naturally equivalent to the usual lifting functor Γ_* .

Proposition 3.6. *The sheafification functor is left-inverse to the lifting functor.*

Proof. We show that $(\hat{\mathcal{E}})^\sim \cong \mathcal{E}$ for any T -equivariant quasi-coherent sheaf on X . The corresponding statement about morphisms then will be evident. With notation as in the proof of Proposition 3.3, we have for every $i \in I$

$$\Gamma(U_i, (\hat{\mathcal{E}})^\sim) = \text{HOM}_{R_i}^{X(G)}(S_i, E_i)_{(0)} = \text{Hom}_{R_i}^{X(G)}((S_i)_{(0)}, E_i) = \text{Hom}_{R_i}^{X(T)}(R_i, E_i) \cong E_i.$$

By naturality, the E_i glue to yield \mathcal{E} . \square

Before we can prove our main result, we need to clarify how homomorphism spaces are related under going back and forth under lifting and sheafification.

Lemma 3.7. (i) *For any $X(G)$ -graded S -module E there exists a natural homomorphism of $X(G)$ -graded S -modules $E \rightarrow \hat{\hat{E}}$.*

(ii) *Let \mathcal{E}, \mathcal{F} be T -equivariant quasi-coherent sheaves on X . Then sheafification induces a surjective homomorphism of K -vector spaces*

$$\text{Hom}_S^{X(G)}(\hat{\mathcal{E}}, \hat{\mathcal{F}}) \twoheadrightarrow \text{Hom}_{\mathcal{O}_X}^T(\mathcal{E}, \mathcal{F}).$$

(iii) *Let E be a $X(G)$ -graded S -module and \mathcal{F} be a quasi-coherent sheaf on X . Then sheafification induces a surjective homomorphism of K -vector spaces*

$$\text{Hom}_S^{X(G)}(E, \hat{\mathcal{F}}) \twoheadrightarrow \text{Hom}_{\mathcal{O}_X}^T(\tilde{E}, \mathcal{F}).$$

Proof. (i) Degree-wise we define a map $\phi_\chi : E_{(\chi)} \rightarrow \text{Hom}_S^{X(G)}(S_{(-\chi)}, E_{(0)})$ for $\chi \in X(G)$ by setting $(\phi_\chi(e))(s) := s \cdot e$ for every $s \in S_{(-\chi)}$. We leave it to the reader to check that this indeed yields a $X(G)$ -homogeneous homomorphism of S -modules.

(ii) By revisiting the constructions of the proof of Proposition 3.6, we conclude that the functorially induced composition

$$\mathrm{Hom}_{\mathcal{O}_X}^T(\mathcal{E}, \mathcal{F}) \longrightarrow \mathrm{Hom}_S^{X(G)}(\widehat{\mathcal{E}}, \widehat{\mathcal{F}}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}^T(\mathcal{E}, \mathcal{F})$$

is a natural isomorphism. In particular, the second homomorphism is surjective.

(iii) By (i) we obtain a homomorphism of S -modules $\mathrm{Hom}_S^{X(G)}(\widehat{\mathcal{E}}, \widehat{\mathcal{F}}) \rightarrow \mathrm{Hom}_S^{X(G)}(E, \widehat{\mathcal{F}})$ which naturally commutes with the maps $\mathrm{Hom}_S^{X(G)}(\widehat{\mathcal{E}}, \widehat{\mathcal{F}}) \rightarrow \mathrm{Hom}_{\mathcal{O}_X}^T(\widetilde{E}, \mathcal{F})$ and $\mathrm{Hom}_S^{X(G)}(E, \widehat{\mathcal{F}}) \rightarrow \mathrm{Hom}_{\mathcal{O}_X}^T(\widetilde{E}, \mathcal{F})$, respectively, which are induced by sheafification. By (ii), the first map is surjective, hence the second must be surjective, too. \square

The can now show our main results, which in particular implies that lifting is left-exact.

Theorem 3.8. *The lifting functor from the category of T -equivariant quasi-coherent sheaves on X to the category of $X(G)$ -graded S -modules is right adjoint to the sheafification functor.*

Proof. We first consider the affine situation and assume that $\hat{X} = \mathrm{Spec}(S)$ and $X = \mathrm{Spec}(R) = \mathrm{Spec}(S_{(0)})$. Denote E an $X(G)$ -graded S -module and F an $X(T)$ -graded (and therefore $X(G)$ -graded, see 2.6) R -module. For simplicity, we write \widehat{F} for the lifting of F . Then we have the isomorphisms of $X(G)$ -graded R -modules

$$\begin{aligned} \mathrm{HOM}_S^{X(G)}(E, \widehat{F}) &= \mathrm{HOM}_S^{X(G)}(E, \mathrm{HOM}_R^{X(G)}(S, F)) \\ &\cong \mathrm{HOM}_R^{X(G)}(E \otimes_S S, F) \cong \mathrm{HOM}_R^{X(G)}(E, F). \end{aligned}$$

Taking invariants with respect to the $X(G)$ -grading, we get

$$\mathrm{HOM}_S^{X(G)}(E, \widehat{F})_{(0)} = \mathrm{HOM}_R^{X(G)}(E, F)_{(0)} = \mathrm{Hom}_R^{X(G)}(E_0, F) = \mathrm{Hom}_R^{X(T)}(E_0, F),$$

where the second equality follows from the fact that F is concentrated in $X(H)$ -degree zero.

For the general case, consider a T -equivariant sheaf \mathcal{F} on X and an $X(G)$ -graded S -module E whose restriction to \hat{X} corresponds to a G -equivariant quasi-coherent sheaf \mathcal{E} . As above, denote $\{U_i\}_{i \in I}$, $\{\hat{U}_i\}_{i \in I}$ a T -invariant (resp. G -invariant) affine cover of X (resp. \hat{X}). The affine case considered before corresponds to isomorphisms $\Gamma(\hat{U}_i, \mathrm{Hom}_{\mathcal{O}_{\hat{U}_i}}^G(\mathcal{E}|_{\hat{U}_i}, \mathcal{F}^G|_{\hat{U}_i})) \rightarrow \Gamma(U_i, \mathrm{Hom}_{\mathcal{O}_{U_i}}^T(\widetilde{E}|_{U_i}, \mathcal{F}|_{U_i}))$ for every $i \in I$. These isomorphisms commute naturally with the restrictions

$$\Gamma(\hat{U}_i, \mathrm{Hom}_{\mathcal{O}_{\hat{U}_i}}^G(\mathcal{E}|_{\hat{U}_i}, \mathcal{F}^G|_{\hat{U}_i})) \rightarrow \Gamma(\hat{U}_i \cap \hat{U}_j, \mathrm{Hom}_{\mathcal{O}_{\hat{U}_i \cap \hat{U}_j}}^G(\mathcal{E}|_{\hat{U}_i \cap \hat{U}_j}, \mathcal{F}^G|_{\hat{U}_i \cap \hat{U}_j}))$$

and

$$\Gamma(U_i, \mathrm{Hom}_{\mathcal{O}_{U_i}}^T(\widetilde{E}|_{U_i}, \mathcal{F}|_{U_i})) \rightarrow \Gamma(U_i \cap U_j, \mathrm{Hom}_{\mathcal{O}_{U_i \cap U_j}}^T(\widetilde{E}|_{U_i \cap U_j}, \mathcal{F}|_{U_i \cap U_j})),$$

respectively for $i, j \in I$. Therefore we obtain an induced homomorphism

$$\mathrm{Hom}_{\mathcal{O}_{\hat{X}}}^G(\mathcal{E}, \mathcal{F}^G) = \Gamma(\hat{X}, \mathrm{Hom}_{\mathcal{O}_{\hat{X}}}^G(\mathcal{E}, \mathcal{F}^G)) \rightarrow \Gamma(X, \mathrm{Hom}_{\mathcal{O}_X}^T(\widetilde{E}, \mathcal{F})) = \mathrm{Hom}_{\mathcal{O}_X}^T(\widetilde{E}, \mathcal{F}).$$

By the naturality of the local isomorphisms and the property that $\mathrm{Hom}_{\mathcal{O}_{\hat{X}}}^G(\mathcal{E}, \mathcal{F}^G)$ is a sheaf it follows that this homomorphism is an isomorphism. It remains to show that $\mathrm{Hom}_S^{X(G)}(E, \widehat{\mathcal{F}}) = \mathrm{Hom}_{\mathcal{O}_W}^G(E, \widehat{\mathcal{F}})$ equals $\mathrm{Hom}_{\mathcal{O}_{\hat{X}}}^G(\mathcal{E}, \mathcal{F}^G)$. For this, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_W}^G(E, \widehat{\mathcal{F}}) & & \\ \phi \downarrow & \searrow \psi & \\ \mathrm{Hom}_{\mathcal{O}_{\hat{X}}}^G(\mathcal{E}, \mathcal{F}^G) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{O}_X}^T(\widetilde{E}, \mathcal{F}), \end{array}$$

where ϕ is the restriction map and ψ the map induced by the sheafification functor. ϕ is injective because $\widehat{\mathcal{F}}$ is an extension of \mathcal{F}^G from \widehat{X} to W and therefore does not have torsion with support on Z . Now, ψ is surjective by Lemma 3.7 (iii), hence both ϕ and ψ are isomorphisms. \square

Remark 3.9. From the proofs of 3.6 and 3.8, it follows that the counit of the adjunction is for every T -equivariant quasicoherent sheaf \mathcal{E} given by the natural map $(\widehat{\mathcal{E}})^\sim \rightarrow \mathcal{E}$ which, using notation from the proof of 3.6, is locally given by the natural isomorphisms $\mathrm{Hom}_{R_i}^{X(T)}(R_i, E_i) \xrightarrow{\cong} E_i$. This is an interesting observation, as it implies that the category of T -equivariant sheaves on X is a reflective localization of the category of $X(G)$ -graded S -modules by the kernel of the sheafification functor. This was previously only known for the case where X is smooth. As was pointed out to me by M. Barakat and M. Lange-Hegermann, this is relevant for current work [BLH12] related to computational toric geometry.

4 Coherence

We have seen in Proposition 3.3 that a torsion free coherent sheaf \mathcal{E} on X lifts to a torsion free coherent sheaf \mathcal{E}^H on \widehat{X} . In this section we want to give similar and refined criteria for the lifting $\widehat{\mathcal{E}}$.

4.1. By Proposition 3.3 (i), properties such as coherence and torsion-freeness do not depend on the additional T -equivariant structure of \mathcal{E} . As our proofs below depend on finding suitable open subsets on X , which must not necessarily be T -invariant, we will therefore consider without loss of generality only the coarser grading by $X(H)$ rather than the $X(G)$ -grading.

Proposition 4.2. *Let D be a Weil divisor on X and denote $\alpha \in X(H) \cong A_{d-1}(X)$ the corresponding class. Then $\widehat{\mathcal{O}(D)} \cong S(\alpha)$. In particular, $\widehat{\mathcal{O}}_X \cong S$.*

Proof. By the isomorphism $\mathcal{O}(D) \cong \mathcal{O}(\alpha)$, we have a decomposition as observed in Remark 3.5:

$$\widehat{\mathcal{O}(D)} \cong \bigoplus_{\beta \in X(H)} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}(\beta), \mathcal{O}(\alpha)) \cong \bigoplus_{\beta \in X(H)} \Gamma(\widehat{X}, \mathcal{O}(\alpha - \beta)) \cong S(\alpha).$$

\square

4.3. By the general properties of good quotients, any open subset U of X can be represented as a good quotient $\widehat{U} // H$, where \widehat{U} is the preimage of U in \widehat{X} under the quotient map. If $U = \mathrm{Spec}(R)$, then from the proof of Propositions 3.3 (ii) and 4.2, we can conclude that $\widehat{U} = \mathrm{Spec}(\widehat{R})$ and $R = \widehat{R}_{(0)}$ with respect to the natural $X(H)$ -grading of \widehat{R} .

Theorem 4.4. *Let \mathcal{E} be a T -equivariant coherent sheaf on X .*

(i) *If \mathcal{E} is torsion free then $\widehat{\mathcal{E}}$ is torsion free and finitely generated.*

(ii) *If \mathcal{E} is reflexive then $\widehat{\mathcal{E}}$ is reflexive and finitely generated.*

Proof. First we note that by the fact that \widehat{X} has codimension 2 in X , coherence (as well as torsion-freeness and reflexivity, respectively, see [Har80, §1]) of \mathcal{E}^H implies that the S -module $\widehat{\mathcal{E}}$ is finitely generated (and torsion free, respectively reflexive). So, assertion (i) follows from Proposition 3.3 (iv).

Now we prove (ii). If \mathcal{E} is reflexive, then by [Har80, Proposition 1.1], we can choose for every point in X a neighbourhood $U = \mathrm{Spec}(R)$ such that there exists a short exact sequence

$$0 \longrightarrow \Gamma(U, \mathcal{E}) \longrightarrow R^n \longrightarrow F \longrightarrow 0,$$

where F is a finitely generated, torsion free R -module. By 4.3, we have $U \cong \hat{U} // H$ with $\hat{U} = \text{Spec}(\hat{R})$ and we can lift this sequence to

$$0 \longrightarrow \Gamma(\hat{U}, \mathcal{E}^H) \longrightarrow \hat{R}^n \longrightarrow G \longrightarrow 0,$$

where G is the homomorphic image of S^n in \hat{F} and therefore torsion-free by Proposition 3.3 (iv). Applying again [Har80, Proposition 1.1], we conclude that \mathcal{E}^G locally reflexive and therefore reflexive. Hence, as the complement of \hat{X} in W has codimension at least two, the module $\hat{\mathcal{E}}$ is reflexive by [Har80, Proposition 1.6]. \square

We will see in Example 5.6 that in general, coherence is not preserved for sheaves with torsion.

5 The case of toric sheaves

We now assume that X is a d -dimensional toric variety with associated fan Δ and $\hat{X} \subseteq \mathbb{Z}^{\Delta(1)} = W$ its standard quotient presentation. As a general reference to toric geometry we refer to [Oda88] and [Ful93]; for specifics of our setting see also [Per11].

5.1. It is customary to denote $M := X(T) \cong \mathbb{Z}^d$ and $\hat{T} := G$, such that $X(G) \cong \mathbb{Z}^{\Delta(1)}$. Moreover, we denote $N = M^*$ and Δ consists of strictly convex polyhedral cones in $N \otimes_{\mathbb{Z}} \mathbb{R}$. We denote $\{l_\rho\}_{\rho \in \Delta(1)}$ the set of primitive vectors of the rays in Δ , which we interpret as linear forms on M . Elements $m \in M$ can be considered as regular functions on T and therefore as rational functions on X . In this case, we write $\chi(m)$, where $\chi(m + m') = \chi(m)\chi(m')$ for any $m, m' \in M$. We have $X(H) \cong A_{d-1}(X)$ and the inclusion of M in to $\mathbb{Z}^{\Delta(1)}$ yields the short exact sequence

$$0 \longrightarrow M \xrightarrow{L} \mathbb{Z}^{\Delta(1)} \longrightarrow A_{d-1}(X) \longrightarrow 0,$$

where L can be represented as a matrix whose rows are formed by the l_ρ . For any strictly convex rational polyhedral cone $\sigma \in \Delta$, we get an affine toric variety U_σ whose M -graded coordinate ring is given by $K[\sigma_M]$ with $\sigma_M = \check{\sigma} \cap M$ and $\check{\sigma}$ denotes the dual cone of σ in $M \otimes_{\mathbb{Z}} \mathbb{R}$. Similarly, we get an exact sequence

$$M \xrightarrow{L_\sigma} \mathbb{Z}^{\sigma(1)} \longrightarrow A_{d-1}(U_\sigma) \longrightarrow 0,$$

where L_σ is the submatrix of L consisting of the rows which correspond to rays in $\sigma(1)$.

We start by recalling some facts about toric sheaves on affine toric varieties and poset representations from [Per04] and [Per11]. Assume that σ is a cone and $S = K[\mathbb{N}^{\sigma(1)}]$ the homogeneous coordinate ring. For any $m, m' \in M$ we write $m \leq_\sigma m'$ if and only if $m' - m \in \sigma_M$. This way we get a preordered set (M, \leq_σ) , which is partially ordered if $\dim \sigma = d$. Equivalently, M becomes a small category, where the morphisms are given by pairs (m, m') whenever $m \leq_\sigma m'$. By the preorder \leq_σ , M also becomes a topological space. Its topology is generated by open sets $U(m) = \{m' \in M \mid m \leq_\sigma m'\}$ for every $m \in M$.

Proposition 5.2 ([Per04, Prop. 5.5] & [Per11, Prop. 2.5]). *The following categories are equivalent:*

- (i) *Toric sheaves on U_σ .*
- (ii) *M -graded $K[\sigma_M]$ -modules.*
- (iii) *Functors from (M, \leq_σ) to the category of K -vector spaces.*
- (iv) *Sheaves of K -vector spaces on M .*

Note that, given a representation E of (M, \leq_σ) , the associated sheaf assigns to any open subset U of M the limit $\varprojlim E_m$, with $m \in U$ (see [Per11, Prop. 2.5]).

Similarly, $\mathbb{N}^{\sigma(1)}$ induces a partial order “ \leq ” on $\mathbb{Z}^{\sigma(1)}$, which is compatible with \leq_σ in the following way.

Lemma 5.3. $L_\sigma(m) \leq L_\sigma(m')$ if and only if $m \leq_\sigma m'$.

Proof. We observe $L_\sigma(m) \leq L_\sigma(m') \Leftrightarrow L_\sigma(m') - L_\sigma(m) \in \mathbb{N}^n \Leftrightarrow l_\rho(m' - m) \geq 0$ for every $\rho \in \sigma(1) \Leftrightarrow m' - m \in \sigma_M$. \square

So, with respect to a fixed cone σ , it is natural to write $m \leq m'$ instead of $L_\sigma(m) \leq L_\sigma(m')$, i.e., $m \leq m'$ if and only if $m \leq_\sigma m'$. Moreover, for every $\underline{c} \in \mathbb{Z}^n$ there exists some $m \in M$ such that $\underline{c} \leq m$. To see this, we observe that we always can choose some $m \in \sigma_M$ with $l_\rho(m) > 0$ for every $\rho \in \sigma(1)$ and some integer $r > 0$ such that $\underline{c} \leq r \cdot m$. So, for every $\underline{c} \in \mathbb{Z}^{\sigma(1)}$ we obtain a nonempty open subset $U_{\underline{c}}$ of M which is given as

$$U_{\underline{c}} = \bigcup_{\underline{c} \leq m} U(m)$$

By Proposition 5.2, every M -graded module E is equivalent to a sheaf of K -vector spaces on M which assigns to every open subset U of M the vector space $E(U) = \varprojlim E_m$, where the limit is taken over all $m \in U$. We use this to define a representation \widehat{E} of $(\mathbb{Z}^{\sigma(1)}, \leq)$ by setting

$$\overline{E}_{\underline{c}} := E(U_{\underline{c}}).$$

By the functoriality of sheaves we have restriction maps $\overline{E}_{\underline{c}} \rightarrow \overline{E}_{\underline{c}'}$ whenever $\underline{c} \leq \underline{c}'$. Hence we obtain a functor from $(\mathbb{Z}^{\sigma(1)}, \leq)$ to the category of K -vector spaces and thus a $\mathbb{Z}^{\sigma(1)}$ -graded S -module $\overline{E} := \bigoplus_{\underline{c} \in \mathbb{Z}^{\sigma(1)}} \overline{E}_{\underline{c}}$ by Proposition 5.2. Clearly this construction is functorial.

Proposition 5.4. Denote $\widehat{E} \cong \bigoplus_{\underline{c} \in \mathbb{Z}^{\sigma(1)}} \widehat{E}_{\underline{c}}$ the $\mathbb{Z}^{\sigma(1)}$ -graded lifting of the sheaf over U_σ associated to E in the sense of Definition 3.4. Then the modules \widehat{E} and \overline{E} are naturally isomorphic. In particular, $\widehat{E}_{\underline{c}} \cong \text{Hom}_{K[\sigma_M]}^M(S_{(\underline{c})}, E)$ is naturally isomorphic to $\overline{E}_{\underline{c}}$ for every $\underline{c} \in \mathbb{Z}^{\sigma(1)}$.

Proof. We write $\underline{c} = (c_\rho \mid \rho \in \sigma(1))$. We can consider $S_{(\underline{c})}$ as an M -graded $K[\sigma_M]$ -submodule of the group ring $K[M]$ with $S_{(\underline{c})} \cong \bigoplus_m K\chi(m)$, where the sum is taken over all $m \in M$ with $l_\rho(m) \geq -c_\rho$. Choose a minimal set of generators s_1, \dots, s_t of $S_{(\underline{c})}$ with degrees m_1, \dots, m_t . Then any M -homogeneous homomorphism is determined by the images of the s_i in the homogeneous components E_{m_i} . Hence, we can identify $\text{Hom}_{K[\sigma_M]}^M(S_{(\underline{c})}, E)$ in a natural way with a subvector space of $\bigoplus_{i=1}^t E_{m_i}$ consisting of tuples (e_1, \dots, e_t) such that $\chi(m - m_i)e_i = \chi(m - m_j)e_j$ whenever $m_i, m_j \leq_\sigma m$. But this vector space has the universal properties of the limit $\varprojlim E_m$ and thus we can naturally identify it with $\widehat{E}_{\underline{c}} = \varprojlim E_m$. The isomorphism of the modules \widehat{E} and \overline{E} then follows from the naturality of this identification. \square

Remark 5.5. By Theorem 3.8 the lifting functor is left-exact and to any toric sheaf \mathcal{E} we can consider its right derived modules

$$\widehat{\mathcal{E}} = \widehat{\mathcal{E}}^{(0)}, \widehat{\mathcal{E}}^{(1)}, \dots$$

By Proposition 5.4 we have now a very nice interpretation of these modules, as we can identify them degree-wise with the right derived functors of the limit functor \varprojlim . Right derived limit functors \varprojlim^i have been pioneered by Roos [Roo61] and have since been studied extensively. Roos also gives a combinatorial analog of the Čech complex which allows in simple cases the explicit computation of the derived functors. We can now understand the left-exactness of the lifting functor combinatorially by the fact that the posets $\{m \in M \mid l_\rho(m) \geq -c_\rho \text{ for all } \rho \in \sigma(1)\}$

$\rho \in \sigma(1)\}$ are not filtered, i.e., for any $m, m' \in U_{\underline{c}}$ there may not exist any $m'' \in U_{\underline{c}}$ with $m'' \leq_{\sigma} m$ and $m'' \leq_{\sigma} m'$, which otherwise would imply the exactness of the limit functor (see [Jen72, Cor. 7.2]).

The following example shows both that lifting in general does not preserve exactness, and the existence of nontrivial right derived modules $\widehat{E}^{(i)}$.

Example 5.6. Let $\sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^3$ be the cone generated by the primitive vectors $l_1 = (1, 0, 0)$, $l_2 = (0, 1, 0)$, $l_3 = (-1, 1, 1)$, $l_4 = (0, 0, 1)$. Denote $\mathfrak{m} \subset K[\sigma_M]$ the maximal homogeneous ideal and consider $K = K[\sigma_M]/\mathfrak{m}$ as a simple module in degree 0. Now for a given $\underline{c} = (c_1, c_2, c_3, c_4) \in \mathbb{Z}^4$, it is straightforward to see that $0 \in M$ is a minimal element in $\{m \in M \mid l_i(m) \geq -c_i\}$ if and only if

$$c_1, c_3 \leq 0, c_2 = c_4 = 0 \quad \text{or} \quad c_1 = c_3 = 0, c_2, c_4 \leq 0.$$

If \underline{c} satisfies one of these conditions, then $\widehat{K}_{\underline{c}} \cong K$ and $\widehat{K}_{\underline{c}} = 0$ otherwise. So there is no lower bound for the c_i such that $\widehat{K}_{\underline{c}}$ vanishes and so \widehat{K} cannot be finitely generated. We observe that \widehat{K} is Artinian and is supported precisely on those torus orbits which get contracted to the fixed point under the quotient map $\mathbf{A}_K^4 \rightarrow U_{\sigma}$.

Moreover, by Lemma 4.2, we have $\widehat{K[\sigma_M]} \cong S$ and the long exact derived sequence of $0 \rightarrow \mathfrak{m} \rightarrow K[\sigma_M] \rightarrow K \rightarrow 0$ starts by:

$$0 \longrightarrow \widehat{\mathfrak{m}} \longrightarrow S \longrightarrow \widehat{K} \longrightarrow \widehat{\mathfrak{m}}^{(1)}.$$

By degree-wise inspection one can see that $\widehat{\mathfrak{m}} = (x_1, x_2, x_3, x_4)$, and therefore $\widehat{\mathfrak{m}}^{(1)}$ cannot be finitely generated as well.

By adjointness, the lifting functor transports injective M -graded $K[\sigma_M]$ -modules to injective $\mathbb{Z}^{\sigma(1)}$ -graded S -modules. In [Per11], codivisorial modules have been considered. For given $\underline{c} \in \mathbb{Z}^{\sigma(1)}$, such a module can be defined as $K[-M^{\underline{c}, I}] = \bigoplus_{m \in -M^{\underline{c}, I}} K\chi(m)$, where I is any subset of $\sigma(1)$ and $M^{\underline{c}, I} = \{m \in M \mid l_{\rho}(m) \geq c_{\rho} \text{ for } \rho \in I\}$. If $\underline{c} = L_{\sigma}(m)$ for some $m \in M$, then $K[-M^{\underline{c}, I}]$ is an injective object in M - $K[\sigma_M]$ -Mod. However, if $K[-M^{\underline{c}, I}]$ is not injective, the following example shows that lifting can exhibit a more bizarre behavior than in the previous example.

Example 5.7. Let $K[\sigma_M]$ be as in Example 5.6 and consider the module $K[-M^{\underline{c}, I}]$ with $\underline{c} = 0$ and $I = \{1, 3\}$. A similar computation as in Example 5.6 shows that $\widehat{K[-M^{\underline{c}, I}]}_{(c_1, 0, c_3, 0)} \cong K^{1-c_1-c_3}$ whenever $c_1 + c_3 \leq 0$. So this module exhibits an infinite family of graded components of any finite dimension. This shows that the lifting functor does not respect combinatorial finiteness in the sense of [Per11].

5.8. Rather than limits, we can also consider colimits associated to representations of (M, \leq_{σ}) . That is, for any M -graded $K[\sigma_M]$ -module E , there is its colimit $\varinjlim E_m$. As the preordered set (M, \leq_{σ}) is filtered, forming the colimit is exact. Given an M -graded $K[\sigma_M]$ -module $E \cong \bigoplus_{m \in M} E_m$, we can associate to it the colimit $\mathbf{E} := \varinjlim E_m$. Similarly, for the lifted S -module \widehat{E} we have the colimit $\widehat{\mathbf{E}} := \varinjlim \widehat{E}_{\underline{c}}$, which is formed over the poset $(\mathbb{Z}^{\sigma(1)}, \leq)$.

Proposition 5.9. *In the above situation we have $\mathbf{E} = \widehat{\mathbf{E}}$.*

Proof. It is easy to see that for every $\underline{c} \in \mathbb{Z}^{\sigma(1)}$ we can find some $m \in M$ such that $\underline{c} \leq m$. Conversely, for every $m \in M$ we can find some $\underline{c} \in \mathbb{Z}^{\sigma(1)}$ such that $m \leq \underline{c}$. It follows that $\varinjlim E_m$ and $\varinjlim \widehat{E}_{\underline{c}}$ are cofinal. \square

If $\dim \sigma < d$, then we have $m \leq_{\sigma} m'$ and $m' \leq_{\sigma} m$ whenever $m' - m \in \sigma_M^{\perp}$. In particular, such a pair (m, m') is an isomorphism in the category M . The following proposition states that,

up to natural equivalence, we do not lose anything essential if we pass from the preordered set (M, \leq_σ) to M/σ_M^\perp with the induced partial order:

Proposition 5.10 ([Per11, Prop. 2.8]). *Let $\Lambda \subseteq \sigma_M$ be a subgroup. Then there is an equivalence of categories between the category of M -graded $K[\sigma_M]$ -modules and the category of M/Λ -graded $K[\sigma_M/\Lambda]$ -modules.*

Note that we state Proposition 5.10 in slightly greater generality than [Per11].

5.11. Now, we are ready to consider the non-affine case. Denote $\{U_\sigma\}_{\sigma \in \Delta}$ the standard covering of X and $\{\hat{U}_\sigma = \text{Spec}(S_\sigma)\}_{\sigma \in \Delta}$ the corresponding cover of \hat{X} given by the preimages of the U_σ . If we take a T -equivariant, i.e., toric sheaf \mathcal{E} on X , we see by Proposition 5.10 that the S_σ -modules $\Gamma(\hat{U}_\sigma, \mathcal{E}^{\hat{T}})$ are naturally equivalent to the lifts of $\Gamma(U_\sigma, \mathcal{E})$ to $K[\mathbb{N}^{\sigma(1)}]$. In particular, it is straightforward to check that coherence, torsion-freeness, and reflexivity are preserved by passing back and forth between $K[\mathbb{N}^{\sigma(1)}]$ and S_σ .

5.12. Given a quasi-coherent sheaf \mathcal{E} on X , we obtain a family of colimits $\mathbf{E}^\sigma := \varinjlim \Gamma(U_\sigma, \mathcal{E})_m$ for $\sigma \in \Delta$. For every pair of faces τ, σ such that τ is a face of σ , the restriction $\Gamma(U_\sigma, \mathcal{E}) \rightarrow \Gamma(U_\tau, \mathcal{E})$ induces a map of directed families over (M, \leq_σ) and (m, \leq_τ) , respectively, and by the universal property of colimits we obtain an induced K -linear isomorphism $\mathbf{E}^\sigma \rightarrow \mathbf{E}^\tau$ (see [Per04, §5.4]). As the face poset of Δ has the zero cone 0 as the unique minimal element, we can use the isomorphisms $\mathbf{E}^\sigma \rightarrow \mathbf{E}^0$ to identify the \mathbf{E}^σ with $\mathbf{E}^0 =: \mathbf{E}$. For the case that \mathcal{E} is coherent, it has been shown in [Per04, §5.4] that $\dim \mathbf{E}$ equals the rank of \mathcal{E} . We can do the same construction for $\hat{\mathcal{E}}$ and obtain a colimit $\hat{\mathbf{E}}$, which, using Proposition 5.9, we can in a natural way identify with \mathbf{E} .

5.13. This construction becomes most interesting for the case that \mathcal{E} (and thus $\hat{\mathcal{E}}$ by Theorem 4.4) is finitely generated and torsion-free. Then the maps $\Gamma(U_\sigma, \mathcal{E})_M \xrightarrow{\cdot \chi(m')} \Gamma(U_\sigma, \mathcal{E})_{m+m'}$ are injective for every $\sigma \in \Delta$, $m \in M$, and $m' \in \sigma_M$. It follows that the induced maps $\Gamma(U_\sigma, \mathcal{E})_m \rightarrow \mathbf{E}$ are injective as well for every $\sigma \in \Delta$ and $m \in M$, and analogously so for the induced maps $\hat{\mathcal{E}}_{\underline{c}} \rightarrow \mathbf{E}$ for $\underline{c} \in \mathbb{Z}^{\Delta(1)}$. This allows a greatly condensed representation of torsion free toric sheaves in terms of families of subvector spaces of a fixed vector space \mathbf{E} which are parameterized by the family of posets $\{(M, \leq_\sigma)\}_{\sigma \in \Delta}$ (see [Per04, Theorem 5.18]).

For the case of reflexive sheaves we have the following structural theorem due to Klyachko.

Theorem 5.14 ([Kly90], [Kly91], see also [Per04]). *The category of coherent reflexive toric sheaves on a toric variety X is equivalent to the category of vector spaces \mathbf{E} endowed with filtrations $0 \subseteq \cdots \subseteq E^\rho(i) \subseteq E^\rho(i+1) \subseteq \cdots \subseteq \mathbf{E}$ for $\rho \in \Delta(1)$ which are full in the sense that $E^\rho(i) = 0$ for $i \ll 0$ and $E^\rho(i) = \mathbf{E}$ for $i \gg 0$.*

5.15. Over U_σ , we observe that for a torsion free $K[\sigma_M]$ -module E we have $\varprojlim E_m$ equals the intersection $\bigcap_{m \leq_\sigma m'} E_{m'}$ in \mathbf{E} . Therefore, given \mathbf{E} and $E^\rho(i)$ for $\rho \in \sigma(1)$ as in Theorem 5.14, one constructs a reflexive module E from this data by setting $E = \bigoplus_{m \in M} E_m$ and $E_m = \bigcap_{\rho \in \sigma(1)} E^\rho(l_\rho(m)) \subseteq \mathbf{E}$.

By Theorem 4.4 we know that for a reflexive toric sheaf \mathcal{E} on X , its lifting $\hat{\mathcal{E}}$ is reflexive as well. The fan $\hat{\Delta}$ associated to \hat{X} in general contains more cones than Δ , but we have a one-to-one correspondence between $\Delta(1)$ and $\hat{\Delta}(1)$ given by, say, $\rho \mapsto \hat{\rho}$. So we know a priori that \mathcal{E} and $\hat{\mathcal{E}}$ are described by the same number of filtrations. The following result shows that these filtrations (in an almost tautological sense) indeed coincide and, moreover, that lifting is indeed “the” correct functor to translate reflexive toric sheaves into $\mathbb{Z}^{\Delta(1)}$ -graded S -modules.

Theorem 5.16. *A toric sheaf \mathcal{E} is coherent and reflexive if and only if $\hat{\mathcal{E}}$ is coherent and*

reflexive. Moreover, if \mathcal{E} and $\widehat{\mathcal{E}}$ are reflexive, then they are canonically described by the same data, i.e., $\widehat{\mathbf{E}} = \mathbf{E}$ and $\widehat{E}^\rho(i) = E^\rho(i)$ for any $\rho \in \Delta(1)$. In particular, lifting induces an equivalence of categories between the category of reflexive toric sheaves on X , the category of reflexive toric sheaves on \widehat{X} , and the category of reflexive $\mathbb{Z}^{\Delta(1)}$ -graded S -modules.

Proof. The statements on coherence and reflexivity follow from Theorem 4.4. It suffices to consider the case that X is affine, i.e., $X = U_\sigma$. So, assume that E is a reflexive M -graded $K[\sigma_M]$ -module, given by filtrations $E^\rho(i)$ of the vector space \mathbf{E} . From this data we can construct a reflexive $\mathbb{Z}^{\sigma(1)}$ -graded S -module F by setting $\mathbf{F} = \mathbf{E}$ and $F^\rho(i) = E^\rho(i)$. Similarly, if we start with the reflexive S -module F , we get a reflexive $K[\sigma_M]$ -module E' by simply identifying the filtrations. We show that $F \cong \widehat{E}$ and $E' = F_{(0)} = E$.

The equality $E' = F_{(0)} = E$ follows from the fact that $E_m = \bigcap_{\rho \in \sigma(1)} E^\rho(l_\rho(m)) = \bigcap_{\rho \in \sigma(1)} F^\rho(l_\rho(m)) = F_m$ (see 5.15), where in the latter equation we identify m with its image $L_\sigma(m) \in \mathbb{Z}^{\sigma(1)}$. Now consider $\widehat{E}_{\underline{c}}$ for some $\underline{c} \in \mathbb{Z}^n$. By 5.15 we have $\widehat{E}_{\underline{c}} = \varprojlim E_m = \bigcap_{\underline{c} \leq m} E_m = \bigcap_{\underline{c} \leq m} \bigcap_{\rho \in \Delta(1)} E^\rho(l_\rho(m)) \subseteq \mathbf{E}$. Now by the fact that the l_ρ are primitive elements in N , we can always choose for any $\rho \in \Delta(1)$ some $m \in M$ such that $l_\rho(m) = c_\rho$. It follows that $\widehat{E}_{\underline{c}} = \bigcap_{\rho \in \Delta(1)} E^\rho(c_\rho) = F_{\underline{c}}$.

For the equivalence of categories, it suffices to remark that for any two reflexive toric sheaves \mathcal{E}, \mathcal{F} , there is a natural bijection $\text{Hom}(\mathcal{E}, \mathcal{F}) \rightarrow \text{Hom}(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$, as any homomorphism of vector spaces $\mathbf{E} \rightarrow \mathbf{F}$ which respects the filtrations also respects any of their intersections. \square

Remark 5.17. For \mathcal{E} reflexive, one can easily show that the S -module $\widehat{\mathcal{E}}$ is isomorphic to $(\Gamma_* \mathcal{E})^\sim$, the reflexive hull of $\Gamma_* \mathcal{E}$. Note that more generally, if \mathcal{E} is torsion free, then $\widehat{\mathcal{E}}$ does not necessarily coincide with $\Gamma_* \mathcal{E}$ modulo torsion.

Remark 5.18. In [Per11], reflexive M -graded $K[\sigma_M]$ -modules have been investigated in terms of the vector space arrangements underlying the associated filtrations. Given such a module E , it is not difficult to see that in general not all possible intersections are realized as the graded components $\Gamma(U_\sigma, \mathcal{E})_m = \bigcap_{\rho \in \sigma(1)} E^\rho(l_\rho(m))$. However, for the vector space arrangement underlying the filtrations associated to $\widehat{\mathcal{E}}$, all possible intersections indeed are realized this way. In this sense, one can consider vector space arrangement in \mathbf{E} underlying the filtrations associated to \widehat{E} as the intersection completion of the vector space arrangement underlying the filtrations associated to E .

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